Quotients of free topological groups

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Reported new results in my talk are based on two joint works
1) Arkady Leiderman, Sidney Morris, Mikhail Tkachenko,
"The Separable Quotient Problem for Topological Groups"
(to appear in Israel J. Math.),
2) Arkady Leiderman, Mikhail Tkachenko,
"Quotients of Free Topological Groups".

For any notions which are not explicitly defined in the talk I advise to consult the monograph of A. Arhangel'skii and M. Tkachenko, *"Topological Groups and Related Structures"*, Chapter 7,
and our recent survey paper (Open Access)
3) Arkady Leiderman and Sidney Morris,
"Separability of topological groups: a survey with open problems", Axioms, 2019, doi:10.3390/axioms8010003.

1. Quotient mappings

All topological spaces and topological groups are assumed to be Hausdorff and all topological spaces are assumed to be infinite.

Assume $\varphi\colon X \to Y$ is a mapping between two topological spaces X and Y such that
1) φ is surjective, 2) φ is continuous, and
3) for $U \subseteq Y$, $\varphi^{-1}(U)$ is open in X implies that U is open in Y.
In this case the mapping φ is called a *quotient* mapping, and we say that Y is a *quotient* of X.

- For instance, if F is a closed subspace of a regular topological space X then a mapping which is the identity on $X \setminus F$ and collapses F to a point is a quotient mapping from X onto a Hausdorff space X/F.
- Every closed mapping and every open mapping between topological spaces is a quotient mapping.
- If $\varphi\colon G \to H$ is a quotient continuous homomorphism from a topological group G onto a topological group H, then φ is an open mapping.

The majority of topological properties are not preserved by quotient mappings. For instance,

- A quotient space of a metric space need not be a Hausdorff space;
- A quotient space of a separable metric space need not have a countable base.

2. The Separable Quotient Problem for Banach Spaces

Let us begin with a famous unsolved problem in Banach space theory. The Separable Quotient Problem for Banach Spaces has its roots in the 1930s and is due to Stefan Banach and Stanisław Mazur.

Problem 2.1

Does every infinite-dimensional Banach space have a quotient Banach space which is separable and infinite-dimensional?

In the literature many special cases of the Separable Quotient Problem for Banach Spaces have been proved, for instance:

Some partial positive results

- Every infinite-dimensional reflexive Banach space has a separable infinite-dimensional quotient Banach space (A. Pełczyński, 1964).
- Every Banach space $C(K)$, where K is a compact space, has a separable infinite-dimensional quotient Banach space (H. Rosenthal, 1969; E. Lacey, 1972).
- Every Banach dual of any infinite-dimensional Banach space, E^*, has a separable infinite-dimensional quotient Banach space (S. Argyros, P. Dodos, V. Kanellopoulos, 2008).

However the general Problem 2.1. for Banach spaces remains unsolved.

Turning to locally convex spaces one can state the analogous problems.

Problem 2.2. (Separable Quotient Problem for Locally Convex Spaces)

Does every infinite-dimensional locally convex space have a quotient locally convex space which is *separable* and infinite-dimensional?

Problem 2.3. (Separable Metrizable Quotient Problem for Locally Convex Spaces)

Does every infinite-dimensional locally convex space have a quotient locally convex space which is *separable and metrizable* and infinite-dimensional?

- Every infinite-dimensional Fréchet space which is non-normable has the separable metrizable topological vector space $\mathbb{R}^\omega$ as a quotient space (M. Eidelheit, 1936).

Efimov spaces and $C_p(X)$

Note that there are many other partial positive solutions in the literature to Problems 2.2, 2.3. In particular, recently positive partial results were obtain for $C_p(X)$ spaces (J. Kąkol, S. Saxon, T. Banakh, W. Sliwa). $C_p(X)$ denotes the space $C(X)$ endowed with the topology of pointwise convergence.

Theorem 2.4. (Jerzy Kąkol, Wieslaw Sliwa, 2018)

Assume there is an infinite compact K such that $C_p(K)$ does not have a quotient LCS which is infinite-dimensional and separable. Then K is an Efimov space, i.e. K contains neither a non-trivial convergent sequence, nor a copy of $\beta\mathbb{N}$.

It is not known whether Efimov compact space exists in ZFC.

Theorem 2.5. (Taras Banakh, Jerzy Kąkol, Wieslaw Sliwa, 2018)

Under $\diamond$ there exists an Efimov compact space K such that $C_p(K)$ does have a quotient LCS which is infinite-dimensional, metrizable (and separable).

However J. Kąkol, S. Saxon and A. Todd (2014) answered Problem 2.2 for locally convex spaces in the negative.
Recall that a *barrel* in a topological vector space is a convex, balanced, absorbing and closed set. A Hausdorff topological vector space E is called barrelled if every barrel in E is a neighborhood of the zero element.

Theorem 2.6.

There exists an infinite-dimensional barrelled locally convex space without any quotient space which is an infinite-dimensional separable locally convex space.

3. A formulation of the Separable Quotient Problem (SQP) for Topological Groups

An abstract group is called *simple* if it has no proper non-trivial normal subgroups.
A topological group G is said to be *topologically simple* if it has no proper non-trivial closed normal subgroups.

Every topologically simple group is either *totally disconnected* or connected.
In every topological space the empty set and the one-point sets are connected; in a totally disconnected space these are the only connected subsets.

Problem 3.1. Infinite SQP for Topological Groups.

Does every non-totally disconnected topological group have a quotient group which is an infinite separable topological group?

Problem 3.2. Infinite Metrizable SQP for Topological Groups.

Does every non-totally disconnected topological group have a quotient group which is an infinite metrizable separable topological group?

Answers

Our answers for Problems 3.1, 3.2.
For topological groups which are closely related to compact groups: Yes.
In a more general setting: No.

One might also reasonably ask: If the topological group G has a quotient group which is infinite and separable, does G necessarily have a quotient group which is infinite, separable and metrizable?
Our answer: No.

Theorem 3.3.

Every infinite σ-compact locally compact group has a quotient group which is an infinite separable metrizable group.

Corollary 3.4.

Every infinite compact group has a quotient group which is an infinite separable metrizable group.

Problem 3.5. Separable Quotient Problem for Locally Compact Groups.

Does every *non-totally disconnected locally compact* group have a separable quotient group which is
(i) non-trivial (not just the identity element);
(ii) infinite;
(iii) metrizable;
(iv) infinite metrizable?

Answer for Problem 3.5.
If in addition G is abelian then: Yes for all items;
In general: Unknown.

4. σ-compact groups, Lindelöf Σ-groups and pseudocompact groups

Recall that the class of Lindelöf Σ-groups contains all σ-compact and all separable metrizable topological groups, and is closed with respect to countable products, closed subgroups and continuous homomorphic images.

Theorem 4.1.

Let G be an infinite Lindelöf Σ-group. Then G has an infinite quotient group with a countable network (hence the quotient group is hereditarily separable).

Corollary 4.2.

Let G be an infinite σ-compact topological group. Then G has an infinite quotient group with a countable network.

However, we find an example of a countable (therefore σ-compact group) precompact abelian group H such that every quotient group of H is either trivial or non-metrizable.

Example 4.3.

For a given prime number p, denote by $\mathbb{C}_p$ the quasicyclic p-group

$$\{z \in \mathbb{T} : z^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$$

considered as a subgroup of the group $\mathbb{T}$. Clearly $\mathbb{C}_p$ is a countable infinite abelian group. Let τ be the *Bohr topology* of $\mathbb{C}_p$, i.e. the maximal precompact topological group topology of $\mathbb{C}_p$. We claim that the group $H = (\mathbb{C}_p, \tau)$ is as required.

A topological space X is called *pseudocompact* if any continuous real-valued function defined on X is bounded. The next result strengthens Theorem 3.4. (for infinite compact groups).

Theorem 4.3.

Every infinite pseudocompact topological group G has a quotient group which is infinite separable compact and metrizable.

Negative answer for precompact groups

Theorem 4.4.

There exists an uncountable zero-dimensional dense subgroup G of the compact abelian group $\mathbb{T}^{\mathfrak{c}}$ such that every countable subgroup of G is closed and every uncountable subgroup of G is dense in G. Hence every quotient group of G is either trivial or non-separable.

5. Definition of the free topological groups

Let X be a Tychonoff space. A topological group $\mathrm{F}(X)$ is called *the (Markov) free topological group* over X if $\mathrm{F}(X)$ satisfies the following conditions:

(i) there is a continuous embedding $\gamma : X \to \mathrm{F}(X)$ such that $\gamma(X)$ algebraically generates $\mathrm{F}(X)$;

(ii) if $f : X \to G$ is a continuous mapping to a topological group G, then there exists a continuous homomorphism $\bar{f} : \mathrm{F}(X) \to G$ such that $f = \bar{f} \circ \gamma$.

Let X be a Tychonoff space. An abelian topological group $\mathrm{A}(X)$ is called *the (Markov) free abelian topological group* over X if $\mathrm{A}(X)$ satisfies the following conditions:

(i) there is a continuous embedding $\gamma : X \to \mathrm{A}(X)$ such that $\gamma(X)$ algebraically generates $\mathrm{A}(X)$;

(ii) if $f : X \to G$ is a continuous mapping to an abelian topological group G, then there exists a continuous homomorphism $\bar{f} : \mathrm{A}(X) \to G$ such that $f = \bar{f} \circ \gamma$.

The free topological groups $\mathrm{F}(X)$, $\mathrm{A}(X)$ always exist and are unique up to isomorphism. In what follows we will identify X with its homeomorphic copy $\gamma(X)$.

Known facts

1. $F(X)$ and $A(X)$ are separable if and only if X is separable;
2. $F(X)$ and $A(X)$ are metrizable if and only if $F(X)$ and $A(X)$ are discrete if and only if X is discrete;
3. $F(X)$ and $A(X)$ are σ-compact if and only if X is σ-compact;
4. $F(X)$ and $A(X)$ are not locally compact or pseudocompact for any infinite X;
5. $A(X)$ is a natural quotient group of $F(X)$;
6. For every X there is a quotient mapping from $A(X)$ onto the group of integers $\mathbb{Z}$.

6. Free topological groups which admit second countable quotient groups

A space X is called *ω-bounded* if the closure of every countable subset of X is compact. Clearly, every compact space is ω-bounded, while every ω-bounded space is countably compact.

Theorem 6.1.

Let X be a non-scattered Tychonoff space. If X has one of the following properties (a) or (b), then both $A(X)$ and $F(X)$ admit an open continuous homomorphism onto the circle group $\mathbb{T}$:

(a) X is normal and countably compact;

(b) X is ω-bounded.

Theorem 6.2.

Let X be an ω-bounded Tychonoff space. Then the following conditions are equivalent:

(a) Every metrizable quotient group of $F(X)$ is discrete and finitely generated.

(b) Every metrizable quotient group of $F(X)$ is finitely generated.

(c) Every metrizable quotient group of $F(X)$ is countable.

(d) X is scattered.

Corollary 6.3.

Let X be either the compact space of ordinals $[0, \alpha]$ with the order topology or the one-point compactification of an arbitrary discrete space. Then every metrizable quotient group of $F(X)$ or $A(X)$ is discrete and finitely generated.

It turns out that the free topological groups on non-pseudocompact zero-dimensional spaces do have non-trivial metrizable quotient groups:

Theorem 6.4.

Let X be a non-pseudocompact zero-dimensional space. Then the groups $F(X)$ and $A(X)$ admit an open continuous homomorphism onto the (countably infinite separable metrizable) discrete group $A(\mathbb{Z})$.

7. Free topological groups which admit quotient groups with a countable network

Theorem 7.1.

Let X be a locally compact or pseudocompact space. Then the groups $A(X)$ and $F(X)$ admit an open continuous homomorphism onto $A(S)$, where S is an infinite compact subspace of the closed unit interval $[0, 1]$, hence $A(S)$ has a countable network.

Theorem 7.2.

Let X be a Lindelöf Σ-space (in particular, σ-compact space). Then the groups $A(X)$ and $F(X)$ admit an open continuous homomorphism onto $A(Y)$, where infinite Y has a countable network, hence $A(Y)$ also has a countable network.

Theorem 8.1.

Let X be a Tychonoff space satisfying the following conditions:

(1) the closure of every countable subset of X is countable and compact;

(2) every countable compact subset of X is a retract of X.

Then every separable quotient group of $F(X)$ is countable.

Corollary 8.2.

Let X be either the space of ordinals $[0, \alpha)$ with the order topology or the one-point compactification of an arbitrary discrete space. Then every separable quotient group of $F(X)$ or $A(X)$ is countable.

9. Open problems

Problem 9.1.

Does there exist an open continuous homomorphism of $A(X)$ onto $A(S)$, where X is an arbitrary Tychonoff space and S is an infinite subspace of the closed unit interval $[0, 1]$?

An even more particular question is open.

Problem 9.2.

Does there exist an open continuous homomorphism of $A(X)$ onto $A(Y)$, where X is an arbitrary Lindelöf space and Y is an infinite space which has a countable network?

R-quotient mappings

A continuous onto mapping $\varphi\colon X \to Y$ is said to be *R-quotient* if for every real-valued function f on Y, the composition $f \circ \varphi$ is continuous iff f is continuous. Clearly, every quotient mapping is R-quotient, but the converse is false.

R-quotient topology

Let $\varphi\colon X \to Y$ be a continuous onto mapping, where the space Y is Tychonoff. Then Y admits the finest topology, say, τ such that the mapping $\varphi\colon X \to (Y, \tau)$ is R-quotient. The topology τ of Y is initial with respect to the family of real-valued functions f on Y such that the composition $f \circ \varphi$ is continuous. The space (Y, τ) is also Tychonoff and τ is finer than the original topology of Y, but the mapping $\varphi\colon X \to (Y, \tau)$ remains continuous.

Theoerem 10.1.

Let $\varphi\colon X \to Y$ be a continuous onto mapping of Tychonoff spaces and $\varphi^*\colon F(X) \to F(Y)$ be an extension of φ to a continuous homomorphism of the free topological groups on X and Y, respectively. Then φ^* is open iff the mapping φ is R-quotient. The same conclusion remains valid for free abelian topological groups.

Proof of Theorem 7.1 for pseudocompact spaces

Let X be an infinite pseudocompact space. Take a countably infinite family of nonempty open sets $\{U_n : n \in \omega\}$ in X such that the closures of all U_n are disjoint and for every $n \in \omega$, pick a point $x_n \in U_n$. For each $n \in \omega$, there is a continuous function $f_n : X \to [0, \frac{1}{2^n}]$ such that $f_n(x_n) = \frac{1}{2^n}$ and $f_n(X \setminus U_n) = 0$. Then the function $f : X \to [0, 1]$ defined by the rule $f(x) = \sum_{n \in \omega} f_n(x)$, for $x \in X$, is continuous. Evidently, $S = f(X)$ is infinite and pseudocompact as a continuous image of the pseudocompact space X. Therefore, S is a compact subset of $[0, 1]$ and f is an R-quotient mapping of X onto S. According to Theorem 10.1 the mapping f extends to an open continuous homomorphism of $A(X)$ onto $A(S)$.

Proof of Theorem 7.2 for σ-compact spaces

Let $X = \bigcup_{n\in\omega} C_n$, where each C_n is a compact subset of X. Take any continuous mapping f from X to the closed unit interval $[0,1]$ such that the image $f(X)$ is infinite. Let τ be a topology on $f(X)$ such that the mapping $f\colon X \to (f(X), \tau)$ is R-quotient. Clearly, τ is finer than the topology of $f(X)$ inherited from $[0,1]$ and the space $Y = (f(X), \tau)$ is Tychonoff. For every $n \in \omega$, consider the compact subspace $K_n = f(C_n)$ of Y. Then K_n admits a continuous one-to-one mapping to $[0,1]$, so K_n is a separable metrizable space. Since $Y = \bigcup_{n\in\omega} K_n$, the space Y has a countable network. Extend f to a continuous onto homomorphism $h\colon A(X) \to A(Y)$. It follows from Theorem 10.1 that the homomorphism h is open. Also, the group $A(Y)$ has a countable network.

A classical theorem of Pelczynski and Semadeni states that for a compact space X, the following conditions are equivalent:

(i) there is no continuous mapping of X onto the segment $[0, 1]$;

(ii) X is scattered.

It is worth noting that the proof of the implication (i) $\Rightarrow$ (ii) presented in the book of Semadeni relies heavily on the facts that a compact Hausdorff space is normal and that a compact scattered space X satisfies $\dim X = 0$.

Let us recall that a space X is *feebly compact* if every infinite family of nonempty open sets in X has a cluster point. It is a well-known fact that in the class of Tychonoff spaces, feeble compactness and pseudocompactness coincide.

Theorem 10.2

Let X be a regular, feebly compact, non-scattered space. Then X contains a closed subspace K which admits a continuous mapping onto the Cantor set $\mathcal{C}$.

Sketch of the proof

Let $\mathcal{P} = \bigcup_{n\in\omega} 2^n$ be the usual binary tree of height ω, where $2 = \{0, 1\}$. Denote by Y a nonempty closed subset of X without isolated points. We can define by induction on n a family $\{U_f : f \in \mathcal{P}\}$ of nonempty open subsets of X satisfying the following conditions for all $f \in \mathcal{P}$:

(i) $U_f \cap Y \neq \emptyset$;

(ii) the sets $\overline{U_{f\frown 0}}$ and $\overline{U_{f\frown 1}}$ are disjoint;

(iii) $\overline{U_{f\frown 0}} \cup \overline{U_{f\frown 1}} \subset U_f$.

We claim that the set $K = \bigcap_{n\in\omega} \bigcup_{f\in 2^n} \overline{U_f}$ is as required.

Identifying the Cantor set $\mathcal{C}$ with 2^ω, one defines a mapping $p\colon K \to \mathcal{C}$ by letting $p(x) = h$ for each $x \in \bigcap_{n\in\omega} \overline{U_{h\restriction n}}$, where $h \in 2^\omega$. Clearly (ii) implies that for every $x \in K$, there exists a unique element $h \in 2^\omega$ with $x \in \bigcap_{n\in\omega} \overline{U_{h\restriction n}}$. Since X is feebly compact, it follows from (iii) that the set $\bigcap_{n\in\omega} \overline{U_{h\restriction n}}$ is nonempty for each $h \in 2^\omega$. Hence $p(K) = \mathcal{C}$. We omit a straightforward verification of the continuity of p which follows from (ii) and (iii).

Thank you!